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Some Results on Linear Rational Trigonometric Interpolation

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Abstract—We present here formulae for calculating the p^{th} derivative of a linear rational trigonometric interpolant written in barycentric form. We give sets of interpolating points for which the interpolant converges exponentially towards the interpolated function. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Trigonometric interpolation, Rational interpolation, Barycentric form, Differentiation, Exponential convergence.

1. INTRODUCTION

Let f be a complex valued function on the real line that is periodic with period 2π, and let φ_k, $k = 0(1)n-1$, be n distinct points of the interval $[0, 2\pi]$, such that $\sigma := \sum_{i=0}^{n-1} \varphi_i \neq 2k\pi$, $\forall\, k \in \mathbb{Z}$. The (balanced [1]) trigonometric polynomial of degree $\leq n/2$ which interpolates the $f_k := f(\varphi_k)$, $k = 0(1)n-1$, between the points φ_k is given by, see [1,2],

$$t[f](\varphi) := \sum_{k=0}^{n-1} f_k \ell_k(\varphi), \tag{1}$$

where we have introduced the trigonometric Lagrange polynomials $\ell_k(\varphi)$,

$$\ell_k(\varphi) := a_k \ell(\varphi) \left(\operatorname{cst}\left(\frac{\varphi - \varphi_k}{2}\right) + c \right),$$

with

$$\ell(\varphi) := \prod_{i=0}^{n-1} \sin\left(\frac{\varphi - \varphi_i}{2}\right), \qquad \operatorname{cst}\varphi := \begin{cases} \csc\varphi, & \text{if } n \text{ is } \mathbf{odd}, \\ \cot\varphi, & \text{if } n \text{ is } \mathbf{even}, \end{cases}$$

and

$$c := \begin{cases} 0, & \text{if } n \text{ is } \mathbf{odd}, \\ \cot\dfrac{\sigma}{2}, & \text{if } n \text{ is } \mathbf{even}. \end{cases}$$

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The a_k are given by

$$(a_k)^{-1} := \prod_{i=0,\, i\neq k}^{n-1} \sin\left(\frac{\varphi_k - \varphi_i}{2}\right), \tag{2}$$

and are called the *weights* of representation (1).

$t[f]$ can be written in *barycentric form* by making use of the relation (which follows by interpolation of $f(\varphi) = 1$)

$$1 = \ell(\varphi) \sum_{k=0}^{n-1} a_k \left(\text{cst}\left(\frac{\varphi - \varphi_k}{2}\right) + c\right). \tag{3}$$

Dividing (1) by (3) and simplifying, we obtain

$$t[f](\varphi) = \frac{\sum_{k=0}^{n-1} a_k \csc\left((\varphi - \varphi_k)/2\right) f_k}{\sum_{k=0}^{n-1} a_k \csc\left((\varphi - \varphi_k)/2\right)}, \tag{4}$$

when the number of interpolating points is **odd** ($n = 2m + 1$), see [1,2].

In the case of an **even** number of interpolating points ($n = 2m$), the barycentric form of the (balanced) trigonometric polynomial is given by [1,3]

$$t[f](\varphi) = \frac{\sum_{k=0}^{n-1} a_k \left(\cot\left((\varphi - \varphi_k)/2\right) + \cot \sigma/2\right) f_k}{\sum_{k=0}^{n-1} a_k \left(\cot\left((\varphi - \varphi_k)/2\right) + \cot \sigma/2\right)}, \tag{5}$$

with σ defined as before.

For equidistant points $\varphi_k = 2k\pi/n$, $k = 0(1)n - 1$, the weights (2) satisfy (after simplification, see [1])

$$a_k = (-1)^k a_0,$$

so that the barycentric formulae (4) and (5) become [4]

$$t[f](\varphi) = \frac{\sum_{k=0}^{n-1} (-1)^k \ \text{cst}\ \left((\varphi - \varphi_k)/2\right) f_k}{\sum_{k=0}^{n-1} (-1)^k \ \text{cst}\ \left((\varphi - \varphi_k)/2\right)}, \tag{6}$$

with cst defined as before. This formula holds for both **odd** and **even** n, as $\text{cst}\,\sigma/2 = 0$. Representation (6) is known to be well conditioned [4].

In order to approximate the derivatives of a function f, we can differentiate the trigonometric polynomial interpolating f. We can either use differentiation matrices or the fast Fourier transform (FFT) in the case of equidistant points. In this case, the first- and second-order differentiation matrices $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ can be given explicitly, see [5,6]. For example, the entries of $\mathbf{D}^{(1)}$ are given by

$$D_{jk}^{(1)} = \begin{cases} \frac{1}{2}(-1)^{j+k} \ \text{cst}\ \left(\frac{\varphi_j - \varphi_k}{2}\right), & \text{if } j \neq k, \\ 0, & \text{if } j = k, \end{cases} \tag{7}$$

and are obtained by differentiating the trigonometric Lagrange polynomials $\ell_k(\varphi)$ at the interpolating points.

The purpose of this article is to generalize these results to the case of linear (in f) rational trigonometric interpolation written in its barycentric form, and to present a rapidly converging rational trigonometric interpolant.

In the first section, we present a general rational trigonometric interpolant written in barycentric form and give formulae for calculating the p^{th} derivative of this rational trigonometric interpolant. In the next section, we propose sets of points for which the rational interpolant converges at the same rate as the trigonometric polynomial interpolating between equidistant points (6). Then, in the final section, we illustrate these results with several numerical examples.

2. DIFFERENTIATION MATRICES FOR RATIONAL FUNCTIONS

The interpolation property of function (6) does not depend on the coefficients appearing in front of the cst $((\varphi - \varphi_k)/2)$. For any numbers $u_k \neq 0$, the function

$$r[f](\varphi) = \sum_{k=0}^{n-1} f_k \tilde{\ell}_k(\varphi) \tag{8}$$

interpolates f between the points φ_k. Here the $\tilde{\ell}_k(\varphi)$ are defined by

$$\tilde{\ell}_k(\varphi) := \frac{u_k \text{ cst}\,((\varphi - \varphi_k)/2)}{\sum_{i=0}^{n-1} u_i \text{ cst}\,((\varphi - \varphi_i)/2)}.$$

The interpolating property of $r[f]$ is independent of the interpolating points φ_k. For any set of different φ_ks, the function defined by (8) with $f_k := f(\varphi_k)$ interpolates f between these φ_ks.

This function is no longer a trigonometric polynomial, but a quotient of two such polynomials, see [7]. When the u_ks do not depend of f, the rational function $r[f]$ is linear in f (i.e., $r[\lambda f + \mu g] = \lambda r[f] + \mu r[g]$, for λ, $\mu \in \mathbb{R}$).

In this section, we give a recurrence relation for calculating the p^{th} derivative of $r[f]$ and generalize formula (7).

2.1. Higher-Order Differentiation Matrices

We present a recurrence relation for calculating the entries of the matrix $\mathbf{D}^{(p+1)}$, $p \geq 0$. We proceed as in [6].

Let us define

$$\alpha(\varphi) := \left(\sum_{i=0}^{n-1} u_i \text{ cst}\left(\frac{\varphi - \varphi_i}{2}\right)\right)^{-1}.$$

Since $u_k \alpha(\varphi) = \tilde{\ell}_k(\varphi) \text{ cst}^{-1}((\varphi - \varphi_k)/2)$, we have (if we differentiate both sides of this equality $p+1$ times)

$$u_k \alpha^{(p+1)}(\varphi) = \sum_{q=0}^{p+1} \binom{p+1}{q} \left(\text{cst}^{-1}\left(\frac{\varphi - \varphi_k}{2}\right)\right)^{(q)} \tilde{\ell}_k^{(p+1-q)}(\varphi). \tag{9}$$

If we evaluate this formula at the interpolating point $\varphi = \varphi_k$, we obtain

$$\begin{aligned} u_k \alpha^{(p+1)}(\varphi_k) &= \sum_{q=0}^{p+1} \binom{p+1}{q} \left(\text{cst}^{-1}\left(\frac{\varphi - \varphi_k}{2}\right)\right)^{(q)}_{|\varphi=\varphi_k} \tilde{\ell}_k^{(p+1-q)}(\varphi_k) \\ &= \sum_{q=0}^{p+1} \binom{p+1}{q} \left(\text{cst}^{-1}\left(\frac{\varphi - \varphi_k}{2}\right)\right)^{(q)}_{|\varphi=\varphi_k} D_{kk}^{(p+1-q)}. \end{aligned} \tag{10}$$

Evaluating (9) at the interpolating point $\varphi = \varphi_j$, $j \neq k$, we get

$$u_k \alpha^{(p+1)}(\varphi_j) = \sum_{q=0}^{p+1} \binom{p+1}{q} \left(\text{cst}^{-1}\left(\frac{\varphi - \varphi_k}{2}\right)\right)^{(q)}_{|\varphi=\varphi_j} D_{jk}^{(p+1-q)}.$$

On the other hand, we know from (10) that (replace k by j)

$$\alpha^{(p+1)}(\varphi_j) = \frac{1}{u_j}\sum_{q=0}^{p+1}\binom{p+1}{q}\left(\mathrm{cst}^{-1}\left(\frac{\varphi-\varphi_j}{2}\right)\right)^{(q)}_{|\varphi=\varphi_j} D_{jj}^{(p+1-q)},$$

so that (for $j \neq k$)

$$\begin{aligned} D_{jk}^{(p+1)} = \mathrm{cst}\left(\frac{\varphi_j-\varphi_k}{2}\right)\Bigg[&\frac{u_k}{u_j}\sum_{q=1}^{p+1}\binom{p+1}{q}\left(\mathrm{cst}^{-1}\left(\frac{\varphi-\varphi_j}{2}\right)\right)^{(q)}_{|\varphi=\varphi_j} D_{jj}^{(p+1-q)} \\ &-\sum_{q=1}^{p+1}\binom{p+1}{q}\left(\mathrm{cst}^{-1}\left(\frac{\varphi-\varphi_k}{2}\right)\right)^{(q)}_{|\varphi=\varphi_j} D_{jk}^{(p+1-q)}\Bigg]. \end{aligned} \tag{11}$$

In the first sum of the right-hand side of (11), we have used the fact that

$$\left(\mathrm{cst}^{-1}\left(\frac{\varphi-\varphi_j}{2}\right)\right)_{|\varphi=\varphi_j} = \begin{cases} \sin\dfrac{\varphi_j-\varphi_j}{2} = 0, & \text{if } n \text{ is } \textbf{odd}, \\ \tan\dfrac{\varphi_j-\varphi_j}{2} = 0, & \text{if } n \text{ is } \textbf{even}. \end{cases}$$

For an **odd** number of interpolating points, we have $\mathrm{cst}^{-1} = \sin$ and we can slightly simplify formula (11) to obtain

$$\begin{aligned} D_{jk}^{(p+1)} = \csc\left(\frac{\varphi_j-\varphi_k}{2}\right)&\sum_{q=0}^{\lfloor p/2\rfloor}\binom{p+1}{2q+1}\frac{(-1)^q}{2^{2q+1}}\left[\frac{u_k}{u_j}D_{jj}^{(p-2q)} - \cos\left(\frac{\varphi_j-\varphi_k}{2}\right)D_{jk}^{(p-2q)}\right] \\ &+\sum_{q=1}^{\lfloor (p+1)/2\rfloor}\binom{p+1}{2q}\frac{(-1)^{q+1}}{2^{2q}}D_{jk}^{(p+1-2q)}, \qquad \text{if } j \neq k. \end{aligned} \tag{12}$$

For an **even** number of interpolating points, we have $\mathrm{cst}^{-1} = \tan$ and we get the formula

$$\begin{aligned} D_{jk}^{(p+1)} = \cot\left(\frac{\varphi_j-\varphi_k}{2}\right)\Bigg[&\frac{u_k}{u_j}\sum_{q=0}^{\lfloor p/2\rfloor}\binom{p+1}{2q+1}\left(\tan\left(\frac{\varphi-\varphi_j}{2}\right)\right)^{(2q+1)}_{|\varphi=\varphi_j} D_{jj}^{(p-2q)} \\ &-\sum_{q=1}^{p+1}\binom{p+1}{q}\left(\tan\left(\frac{\varphi-\varphi_k}{2}\right)\right)^{(q)}_{|\varphi=\varphi_j} D_{jk}^{(p+1-q)}\Bigg], \qquad \text{if } j \neq k. \end{aligned} \tag{13}$$

($\lfloor\,\rfloor$ denotes floor integer value.)

For the diagonal elements of the differentiation matrix, we use the relation $\sum_{k=0}^{n-1}\tilde{\ell}_k(\varphi) = 1$ (compare with (3)) and get

$$\sum_{k=0}^{n-1}\tilde{\ell}_k^{(p+1)}(\varphi) = 0, \qquad \forall\,\varphi \text{ and } p = 0,1,2,\ldots, \tag{14}$$

so that for $D_{jj}^{(p+1)}$, we obtain

$$D_{jj}^{(p+1)} = -\sum_{k=0,\,k\neq j}^{n-1} D_{jk}^{(p+1)}. \tag{15}$$

Note that, for $p = 0$, $\mathbf{D}^{(0)}$ is the identity matrix and formulae (12) and (13) give

$$D_{jk}^{(1)} = \begin{cases} \dfrac{1}{2}\dfrac{u_k}{u_j}\,\mathrm{cst}\left(\dfrac{\varphi_j-\varphi_k}{2}\right), & \text{if } j \neq k, \\ -\displaystyle\sum_{i=0,\,i\neq j}^{n-1} D_{ji}^{(1)}, & \text{if } j = k. \end{cases} \tag{16}$$

REMARKS.

1. When $u_k = (-1)^k$, $k = 0(1)n-1$, and the points are equidistant formula (16) becomes formula (7).
2. We can use formulae (12)–(16) for the trigonometric polynomial interpolating between arbitrary points. All we need to do is set $u_k = a_k$ as defined in (2).

3. Relation (14) has already been used in the polynomial case, see, for example, [8–10]. As we shall see in Section 4 (and [11] for more details) formula (16) is less sensitive to rounding errors than the classical formula (7).

For equidistant points, Welfert [12] has shown that the differentiation matrix $\mathbf{D}^{(p+1)}$ of the trigonometric polynomial is related to the $(p+1)^{\text{th}}$-power of $\mathbf{D}^{(1)}$. For n **odd**, we simply have

$$\mathbf{D}^{(p+1)} = \left(\mathbf{D}^{(1)}\right)^{p+1}.$$

For n **even**, Welfert has shown

$$\mathbf{D}^{(2s+1)} = \left(\mathbf{D}^{(1)}\right)^{2s+1},$$

$$\mathbf{D}^{(2s)} = \left(\mathbf{D}^{(1)}\right)^{2s} + \frac{(-1)^s (n/2)^{2s-1}}{2}\mathbf{x}\mathbf{x}^{\mathsf{T}},$$

with the n-vector $\mathbf{x} = (1,-1,1,-1,\dots,1,-1)^{\mathsf{T}}$.

For arbitrary points (which are not a translation of equidistant points), the differentiation matrix $\mathbf{D}^{(p+1)}$ of the trigonometric polynomial is given by the relation

$$\mathbf{D}^{(p+1)} = \left(\mathbf{D}^{(1)}\right)^{p+1}.$$

Sometimes it can be useful to know derivatives of the interpolating function (8) at points that are not interpolating points (for example, if one wants to use the Newton-Raphson method). We can find a general formula for this purpose.

Let $\xi \neq \varphi_k$ $(k = 0(1)n-1)$ be an arbitrary point where we want to calculate the p^{th} derivative of $r[f]$. Let us rewrite (8) as

$$r[f](\varphi)\alpha^{-1}(\varphi) = \sum_{k=0}^{n-1} u_k f_k \operatorname{cst}\left(\frac{\varphi-\varphi_k}{2}\right),$$

with $\alpha(\varphi)$ defined as before. If we differentiate (p times) both sides of the preceding equation and evaluate at the point ξ, we get the recursive formula

$$r[f]^{(p)}(\xi) = \frac{\displaystyle\sum_{q=0}^{p-1}\binom{p}{q}\sum_{k=0}^{n-1} u_k \left(\operatorname{cst}\left((\varphi-\varphi_k)/2\right)\right)^{(p-q)}_{|\varphi=\xi}\left(f_k - r[f](\varphi)\right)^{(q)}_{|\varphi=\xi}}{\displaystyle\sum_{k=0}^{n-1} u_k \operatorname{cst}\left((\xi-\varphi_k)/2\right)}.$$

3. AN EXPONENTIALLY CONVERGENT LINEAR RATIONAL TRIGONOMETRIC INTERPOLANT

In [13], a linear rational interpolant is presented that converges exponentially towards the interpolated function if the latter is analytic in an ellipse. We generalize this result to trigonometric rational interpolation.

In Theorem 3.1 of [7], Berrut presented a 2π-periodic rational trigonometric function that interpolates a function f between arbitrary points φ_k and does not have any pole on the circle. This rational function is given by

$$r[f](\varphi) = \frac{\displaystyle\sum_{k=0}^{n-1}(-1)^k \operatorname{cst}\left((\varphi-\varphi_k)/2\right) f_k}{\displaystyle\sum_{k=0}^{n-1}(-1)^k \operatorname{cst}\left((\varphi-\varphi_k)/2\right)}. \tag{17}$$

We can see in Table 2 of [7] and in columns labelled random in Tables 1 and 2 of Section 4, that for arbitrary interpolating points, the rational function (17) does not converge rapidly to the interpolated function.

We propose here sets of interpolating points for which the fast convergence of the trigonometric polynomial interpolating between equidistant points is preserved. The idea is to take the trigonometric polynomial interpolating f between equidistant points $\theta_k := 2k\pi/n$, $k = 0(1)n-1$, written in barycentric form $t[f](\theta)$, and then move the points θ_k by a conformal map ($\theta_k \to \varphi_k$). The resulting function is then a rational trigonometric interpolant with the same rate of convergence as the trigonometric polynomial.

3.1. Construction of the Trigonometric Rational Interpolant

Let I,J be two intervals in $\mathbb{R}$. Let g be a 2π-periodic map such that $g(J) = I$ and let f be a 2π-periodic function defined on the interval I. Without loss of generality, set $J = I = [0, 2\pi]$.

To prove the convergence of the rational function, we proceed as in [13]. We need the function of two variables [14] defined by

$$w(\psi,\theta) := \frac{\text{cst}\left(\left(g(\psi) - g(\theta)\right)/2\right)}{\text{cst}\left((\psi - \theta)/2\right)}. \tag{18}$$

We want to interpolate this function. For this purpose, we freeze the variable ψ and construct the following trigonometric polynomial $t[w](\psi,\theta)$ between the n equidistant points $\theta_k = 2k\pi/n$, $k = 0(1)n-1$:

$$t[w](\psi,\theta) = a_0\ell(\theta)\sum_{k=0}^{n-1}(-1)^k \text{ cst}\left(\frac{\theta-\theta_k}{2}\right) w(\psi,\theta_k). \tag{19}$$

When $\psi = \theta \neq \theta_k$, we define for den,

$$\text{den}[w](\theta) := t[w](\theta,\theta) = a_0\ell(\theta)\sum_{k=0}^{n-1}(-1)^k \text{ cst}\left(\frac{g(\theta)-g(\theta_k)}{2}\right). \tag{20}$$

If we repeat the same construction for the function $f(g(\theta))w(\psi,\theta)$, we define for num when $\psi = \theta \neq \theta_k$,

$$\text{num}[(f\circ g)w](\theta) := a_0\ell(\theta)\sum_{k=0}^{n-1}(-1)^k \text{ cst}\left(\frac{g(\theta)-g(\theta_k)}{2}\right) f(g(\theta_k)). \tag{21}$$

By forming the quotient of (21) and (20), we get an interpolating function of $f\circ g$ $(= ((f\circ g)w)/w)$ written in barycentric form

$$r[f\circ g](\theta) := \frac{\text{num}[(f\circ g)w](\theta)}{\text{den}[w](\theta)} = \frac{\displaystyle\sum_{k=0}^{n-1}(-1)^k \text{ cst}\left((g(\theta)-g(\theta_k))/2\right)f(g(\theta_k))}{\displaystyle\sum_{k=0}^{n-1}(-1)^k \text{ cst}\left((g(\theta)-g(\theta_k))/2\right)}.$$

If we set $\varphi := g(\theta)$ and $\varphi_k := g(\theta_k)$, this is precisely the rational function (17) for transformed points. We use this quotient of two polynomial interpolating functions to prove the convergence of the rational function.

3.2. Convergence of the Rational Trigonometric Interpolant

In order to prove the convergence of the linear rational trigonometric interpolant (17), we need the following results taken from [4,15]. Here, Π is the class of periodic, complex valued functions of $\mathbb{R}$ with period 2π.

THEOREM 1. *If $f \in \Pi$ is represented by the absolutely convergent Fourier series*

$$f(\varphi) = \sum_{k=-\infty}^{\infty} c_k e^{ik\varphi}, \qquad c_k := \int_0^{2\pi} f(\tau) e^{-ik\tau}\, d\tau,$$

and if $t[f]$ is the unique (and, if n is even, balanced) trigonometric polynomial interpolating f between the sampling points $\varphi_k = 2k\pi/n$, $k = 0(1)n-1$, given by (6), then we have for all real φ,

$$|f(\varphi) - t[f](\varphi)| \le 2 \sum_{|k| \ge m}{}' |c_k|, \tag{22}$$

where $m := \lfloor n/2 \rfloor$ and the prime is taken to mean that, for n even, the factor $1/2$ is to be inserted in the terms $k = \pm m$.

If f has simple jump discontinuities in the p^{th} derivative, then estimate (22) enables us to prove [4]

$$|f(\varphi) - t[f](\varphi)| \le \frac{2K}{m^p}\left(\frac{1}{m} + \frac{2}{p}\right),$$

with $K > 0$ a constant (depending on the function f).

If f is analytic in the strip $S_a := \{\eta : |\operatorname{Im} \eta| \le a\}$, where $a > 0$ and $|f(\varphi)| \le M$, we then have [15]

$$|f(\varphi) - t[f](\varphi)| \le 2M \cot\frac{a}{2} e^{-an}. \tag{23}$$

We now come to the rate of convergence of the linear rational trigonometric interpolant (17).

THEOREM 2. *Let $g \in \Pi$ be a map such that $g(J) = I$ and such that w defined in (18) is bounded and analytic in $S_{a_1} \times S_{a_1}$ ($S_{a_1} := \{\eta : |\operatorname{Im}\eta| \le a_1\}$, $a_1 > 0$). Let f be a function such that $f \circ g \in \Pi$. Let $r[f](\varphi) \equiv r[f \circ g](\theta)$, $\varphi = g(\theta)$, be the rational function (17) interpolating f between the transformed points $\varphi_k := g(\theta_k)$. Then, for every $\varphi \in I$, we have the following.*

- *If $f \circ g$ has simple jump discontinuities in the p^{th} derivative, then*

$$|f(\varphi) - r[f](\varphi)| = \mathcal{O}\left(n^{-p}\right). \tag{24}$$

- *If $f \circ g$ is analytic in a strip $S_{a_2} := \{\eta : |\operatorname{Im}\eta| \le a_2\}$, where $a_1 \ge a_2 > 0$ and $|f(\varphi)| \le M_2$, then*

$$|f(\varphi) - r[f](\varphi)| = \mathcal{O}\left(\rho^{-n}\right), \qquad \textit{with } \rho := e^{-a_2}. \tag{25}$$

PROOF. We only show equation (24). Equation (25) is obtained in the same manner. Using (23), we have $\forall\, \psi \in S_{a_1}$, as w is in $\Pi \times \Pi$,[1]

$$|w(\psi, \theta) - t[w](\psi, \theta)| \le 2M_1 \cot\frac{a_1}{2} e^{-a_1 n},$$

where $|w(\psi, \theta)| \le M_1$, $\forall\, (\psi, \theta) \in S_{a_1} \times S_{a_1}$ and $t[w](\psi, \theta)$ is the trigonometric polynomial interpolating w between the points θ_k given by (19). This yields a bound for the error of the interpolant $\text{den}[w](\theta)$:

$$|w(\theta, \theta) - \text{den}[w](\theta)| = \mathcal{O}(\rho^{-n}), \quad \text{with } \rho := e^{-a_1}.$$

[1]For n **odd**, w is 4π-periodic in both variables. In this case, we can use two changes of variables so that w is again 2π-periodic in both variables.

If $f \circ g$ has simple jump discontinuities in the p^{th} derivative, then we find

$$|(f \circ g)(\theta)w(\theta,\theta) - \text{num}[(f \circ g)w](\theta)| = \mathcal{O}\left(n^{-p}\right).$$

Since g, analytic on S_{a_1}, is continuous on S_{a_1}, we have $\min_{\theta \in S_{a_1}} |w(\theta,\theta)| > 0$. Therefore, we can divide the numerator and the denominator by $w(\theta,\theta)$, which yields

$$\begin{aligned} r\left[f \circ g\right](\theta) = \frac{\text{num}[(f \circ g)w](\theta)}{\text{den}[w](\theta)} &= \frac{(f \circ g)(\theta)w(\theta,\theta) + \mathcal{O}\left(n^{-p}\right)}{w(\theta,\theta) + \mathcal{O}\left(\rho^{-n}\right)} \\ &= \frac{(f \circ g)(\theta) + \mathcal{O}\left(n^{-p}\right)}{1 + \mathcal{O}(\rho^{-n})} \\ &= (f \circ g)(\theta) + \mathcal{O}\left(n^{-p}\right). \end{aligned}$$

∎

Theorem 2 is a generalization of Theorem 4 presented in [13]. In fact, as mentioned in [7], the Čhebyšhev interpolating polynomial on $[-1,1]$ is simultaneously an interpolating cosine polynomial on the upper half circle. The problem can be bijectively mapped on the upper half circle by the function $\phi = \arccos x$.

4. NUMERICAL EXAMPLES

In this section, we present two numerical examples for the interpolation of a function and the approximation of its first derivative by the trigonometric polynomial (6) and the rational function written in barycentric form (17), for various sets of interpolating points φ_k, $k = 0(1)n-1$.

All the computations have been done on a Sun Ultra 5, using Matlab 5.3. In the following tables, "Equally" means that we use the polynomial interpolating between the equidistant points $\varphi_k = (2k\pi)/n$, $k = 0(1)n-1$. "Random" means that we use random points (generated by the MATLAB routine "rand") as interpolating points for the rational function (17). We also reproduce the results for Čhebyšhev points of the second kind given in [7], denoted "T-Points" in the following. In this case, the points are shifted toward the boundary of the interval $[0,2\pi]$. If we want to shift the points toward the interior of the interval, we can use the Kosloff and Tal-Ezer shift [16] on the circle. The transform is given by

$$g(\varphi) := g(\varphi,\alpha) := \arccos\left(\frac{\arcsin(\alpha\cos(\varphi))}{\arcsin(\alpha)}\right), \qquad \varphi \in [0,2\pi], \quad \alpha \in (0,1).$$

When $\alpha \to 0$, the points remain equidistant on the circle, whereas when $\alpha \to 1$, the points are shifted toward the interior of the domain and the function $g(\varphi)$ has singularities that come close to the interval of interpolation $[0,2\pi]$. In the limit case ($\alpha = 1$), the exponential convergence is lost, see [13] or [16] for details.

For the Čhebyšhev and Kosloff and Tal-Ezer map, the function of two variables defined in (18) is not analytic. In this case, we define the function $\tilde{w}$ by $\tilde{w}(\psi,\theta) := w(\psi,\theta)(g'(\psi)/(g'(\psi)+c_0))$, with $c_0 > 0$ a constant such that $\tilde{w}$ is analytic in $S_{a_1} \times S_{a_1}$. Section 3.1 (and Theorem 2) can then be carried out with $\tilde{w}$ instead of w.

Two test functions have been interpolated by the four different interpolating functions. To measure the interpolation error, we consider the 2000 equidistant points

$$\tilde{\varphi}_k := \frac{k\pi}{1000}, \qquad k = 0,\ldots,1999,$$

in the interval $[0,2\pi]$ and calculate the maximum absolute error at the points $\tilde{\varphi}_k$. The results are presented in Tables 1 and 2.

In Table 1, we clearly see the exponential convergence of the trigonometric polynomial (6) and the rational function (17) for transformed points ("T-Points" and "KT-Points" with $\alpha = 0.9$)

Table 1. Interpolation errors for $f(\varphi) = 3/(2 + \cos(\varphi))$.

n	Equally	Random	T-Points	KT-Points (0.9)
10	$2.74 \cdot 10^{-3}$	$6.12 \cdot 10^{-1}$	$2.60 \cdot 10^{-2}$	$5.62 \cdot 10^{-4}$
20	$3.81 \cdot 10^{-6}$	$1.94 \cdot 10^{-1}$	$4.82 \cdot 10^{-4}$	$2.50 \cdot 10^{-7}$
50	$1.18 \cdot 10^{-14}$	$1.29 \cdot 10^{-1}$	$2.42 \cdot 10^{-9}$	$6.93 \cdot 10^{-14}$
100	$4.00 \cdot 10^{-15}$	$3.91 \cdot 10^{-2}$	$5.77 \cdot 10^{-15}$	$4.89 \cdot 10^{-15}$
200	$5.77 \cdot 10^{-15}$	$1.43 \cdot 10^{-2}$	$6.66 \cdot 10^{-15}$	$9.33 \cdot 10^{-15}$
500	$8.88 \cdot 10^{-15}$	$1.78 \cdot 10^{-3}$	$9.33 \cdot 10^{-15}$	$1.07 \cdot 10^{-14}$
1000	$8.88 \cdot 10^{-15}$	$3.27 \cdot 10^{-4}$	$1.51 \cdot 10^{-14}$	$1.78 \cdot 10^{-14}$

Table 2. Interpolation errors for $f(\varphi) = 1/(1 + 25\cos^2(\varphi))$.

n	Equally	Random	T-Points	KT-Points (0.9)
10	$6.39 \cdot 10^{-1}$	$5.32 \cdot 10^{-1}$	$6.66 \cdot 10^{-1}$	$4.21 \cdot 10^{-1}$
20	$1.32 \cdot 10^{-1}$	$5.73 \cdot 10^{-1}$	$3.46 \cdot 10^{-1}$	$1.88 \cdot 10^{-1}$
50	$1.37 \cdot 10^{-2}$	$5.28 \cdot 10^{-1}$	$4.24 \cdot 10^{-2}$	$1.29 \cdot 10^{-2}$
100	$4.62 \cdot 10^{-5}$	$3.21 \cdot 10^{-2}$	$1.23 \cdot 10^{-3}$	$5.27 \cdot 10^{-5}$
200	$2.26 \cdot 10^{-9}$	$5.92 \cdot 10^{-2}$	$7.84 \cdot 10^{-7}$	$2.90 \cdot 10^{-9}$
500	$2.00 \cdot 10^{-15}$	$8.70 \cdot 10^{-3}$	$3.33 \cdot 10^{-15}$	$5.11 \cdot 10^{-15}$
1000	$2.66 \cdot 10^{-15}$	$9.04 \cdot 10^{-3}$	$5.33 \cdot 10^{-15}$	$5.11 \cdot 10^{-15}$

toward the function $f(\varphi) = 3/(2 + \cos(\varphi))$. In the case of arbitrary points ("Random"), we see a slow convergence, as expected.

In Table 2, we can see the same behavior as in Table 1. However, for small n ($n \leq 50$), we cannot see the exponential improvement of the interpolant.

In Tables 3 and 4, we have the approximation errors for the first derivative of the two test functions at the interpolating points. We have added a column to compare the results obtained with formulae (7) and (16) for the trigonometric polynomial interpolating between equidistant points (6). To differentiate the rational function (17), we use formula (16).

As expected, the approximation errors for the derivative of the first test function are good, except for the random interpolating points.

It is interesting to note that formula (16) (with relation (15) for the diagonal elements of the matrix $\mathbf{D}^{(1)}$) leads to better results than using the classical formula (7) for $n \geq 50$. In Table 3,

Table 3. Approximation errors for the derivative of $f(\varphi) = 3/(2 + \cos(\varphi))$.

n	Equally (7)	Equally (16)	Random	T-Points	KT-Points (0.9)
10	$1.35 \cdot 10^{-2}$	$1.35 \cdot 10^{-2}$	$3.00 \cdot 10^{-1}$	$9.91 \cdot 10^{-2}$	$3.14 \cdot 10^{-3}$
20	$3.79 \cdot 10^{-5}$	$3.79 \cdot 10^{-5}$	$2.91 \cdot 10^{-1}$	$3.36 \cdot 10^{-3}$	$2.79 \cdot 10^{-6}$
50	$3.06 \cdot 10^{-13}$	$2.50 \cdot 10^{-13}$	$2.25 \cdot 10^{-1}$	$4.08 \cdot 10^{-8}$	$1.92 \cdot 10^{-12}$
100	$5.89 \cdot 10^{-13}$	$2.33 \cdot 10^{-14}$	$5.52 \cdot 10^{-2}$	$2.27 \cdot 10^{-13}$	$2.29 \cdot 10^{-14}$
200	$2.05 \cdot 10^{-12}$	$3.77 \cdot 10^{-14}$	$1.16 \cdot 10^{-1}$	$1.36 \cdot 10^{-12}$	$6.68 \cdot 10^{-14}$
500	$2.04 \cdot 10^{-11}$	$1.36 \cdot 10^{-13}$	$1.86 \cdot 10^{-1}$	$1.60 \cdot 10^{-11}$	$1.47 \cdot 10^{-13}$
1000	$7.60 \cdot 10^{-11}$	$2.51 \cdot 10^{-13}$	$1.45 \cdot 10^{-1}$	$3.24 \cdot 10^{-11}$	$3.86 \cdot 10^{-13}$

Table 4. Approximation errors for the derivative of $f(\varphi) = 1/(1 + 25\cos^2(\varphi))$.

n	Equally (7)	Equally (16)	Random	T-points	KT-Points (0.9)
10	$8.97 \cdot 10^{-1}$	$8.97 \cdot 10^{-1}$	$2.44 \cdot 10^{0}$	$1.46 \cdot 10^{0}$	$2.54 \cdot 10^{0}$
20	$1.19 \cdot 10^{0}$	$1.19 \cdot 10^{0}$	$2.26 \cdot 10^{0}$	$2.58 \cdot 10^{0}$	$2.05 \cdot 10^{0}$
50	$3.10 \cdot 10^{-1}$	$3.10 \cdot 10^{-1}$	$1.89 \cdot 10^{0}$	$8.65 \cdot 10^{-1}$	$3.09 \cdot 10^{-1}$
100	$2.33 \cdot 10^{-3}$	$2.33 \cdot 10^{-3}$	$1.94 \cdot 10^{0}$	$4.67 \cdot 10^{-2}$	$2.74 \cdot 10^{-3}$
200	$2.26 \cdot 10^{-7}$	$2.26 \cdot 10^{-7}$	$1.24 \cdot 10^{0}$	$5.63 \cdot 10^{-5}$	$3.01 \cdot 10^{-7}$
500	$9.66 \cdot 10^{-12}$	$3.11 \cdot 10^{-14}$	$4.41 \cdot 10^{-1}$	$3.83 \cdot 10^{-13}$	$3.33 \cdot 10^{-14}$
1000	$3.77 \cdot 10^{-9}$	$7.37 \cdot 10^{-14}$	$4.62 \cdot 10^{-1}$	$1.50 \cdot 10^{-12}$	$8.79 \cdot 10^{-14}$

for $n = 1000$, we gain two powers of 10 in the approximation of the derivative. The reason for this has already been given in [9–11]. Keeping relation (15) is better than calculating more precisely some elements of the matrix $\mathbf{D}^{(1)}$. Note that to alleviate "smearing" (see [4] for details on this last concept), we have rearranged the sum of the row from the smallest element to the largest (in absolute value), as suggested in [9–11].

In Table 4, we can see the same behavior as in Table 3, but it takes larger n to get small errors. The approximation error is larger for the second test function. Again, using relation (15) leads to a significant improvement in the approximation of the derivative of the function when the major part of the approximation errors are due to roundoff. For $n = 1000$, we gain five powers of 10 in the approximation of the first derivative.

5. CONCLUSION

We have presented here formulae for calculating the p^{th} derivative of a linear rational trigonometric interpolant written in barycentric form.

We also have proposed sets of interpolating points for which the rational function converges at the same rate toward the interpolated function as the trigonometric polynomial. The rational function could be used to approximate the "inverse" of a periodic function. Like in [17], we could invert the boundary correspondence function in numerical conformal mapping.

In the last section, we have presented numerical results and we have seen that for calculating precisely the derivative of the trigonometric polynomial, we should rather use formula (16) (i.e., (15) for the diagonal elements) than formula (7).

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